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L^p-Theory of the Stokes equation in a half space

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Abstract. In this paper, we investigate L^p-estimates for the solution of the Stokes equation in a half space H where $1 \leq p \leq \infty$. It is shown that the solution of the Stokes equation is governed by an analytic semigroup on $BUC_\sigma(H)$, $C_{0,\sigma}(H)$ or $L^\infty_\sigma(H)$. From the operatortheoretical point of view it is a surprising fact that the corresponding result for $L^1_\sigma(H)$ does not hold true. In fact, there exists an L^1-function f satisfying $div f = 0$ such that the solution of the corresponding resolvent equation with right hand side f does not belong to L^1. Taking into account however a recent result of Kozono on the nonlinear Navier-Stokes equation, the L^1-result is not surprising and even natural. We also show that the Stokes operator admits a R-bounded H^∞-calculus on L^p for $1 < p < \infty$ and obtain as a consequence maximal L^p–L^q-regularity for the solution of the Stokes equation.

1. Introduction

In this paper, we consider the Stokes equation in a half space $H := \mathbb{R}^{n+1}_+$, i.e. we consider the set of equations

$$\begin{aligned} u_t - \Delta u + \nabla p &= f \quad \text{in } H \times (0,\infty) \\ \operatorname{div} u &= 0 \quad \text{in } H \times (0,\infty) \\ u &= 0 \quad \text{on } \partial H \times (0,\infty) \\ u(0) &= u_0 \end{aligned} \tag{1.1}$$

where $u = (u_1, \ldots, u_n, u_{n+1})^T$ is interpreted as the velocity field and p as the pressure. We are interested in the L^p-theory of the solution of (1.1) where $1 \leq p \leq \infty$. If $1 < p < \infty$, one usually defines the subspace $L^p_\sigma(H)$ of $L^p(H)$ consisting of all solenoidal functions f in L^p and associates to (1.1) the so-called Stokes operator A in $(L^p_\sigma(H))^{n+1}$ defined as

$$\begin{aligned} Au &:= P\Delta u \\ D(A) &:= (W^{2,p}(H) \cap W^{1,p}_0(H) \cap L^p_\sigma(H))^{n+1}. \end{aligned}$$

Here P denotes the Helmholtz projection in $L^p(H)$. Then it is well known that the Stokes operator A generates an analytic semigroup on $L^p_\sigma(H)$ for $1 < p < \infty$ (see for instance

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[Sol77], [McC81], [Uka87]). It seems to be unknown whether system (1.1) is also solvable in $L^1(H)$ or $L^\infty(H)$. Observe that the Helmholtz projection is not bounded in $L^1(H)$ or $L^\infty(H)$. Thus the usual decomposition of $L^p(H)$ in $L^p_\sigma(H)$ and its orthogonal complement which is true for $1 < p < \infty$ is no longer possible if $p = 1$ or $p = \infty$.

The strategy we use to solve the Stokes system and to get sharp regularity results on the resolvent is the following: Taking Fourier transforms we obtain a representation of the solution of the corresponding resolvent equation as a sum of two terms, the first being the resolvent of the Dirichlet Laplacian on $L^p(\mathbb{R}^{n+1}_+)$, the second one being a remainder term. In Section 3 we derive pointwise upper bounds on the remainder term which allow to prove L^∞-estimates for the solution of the corresponding resolvent problem, i.e. we obtain estimates of the form

$$\|u\|_\infty \leq \frac{M}{|\lambda|}\|f\|_\infty$$

for the equation defined below in (2.1), where λ belongs to a suitable sector in the complex plane.

The above estimate allows us to prove that the Stokes operator generates an analytic semigroup on the spaces $BUC_\sigma(H)$, $C_{0,\sigma}(H)$, $L^\infty_\sigma(H)$, which is, of course, not strongly continuous in the latter case.

The case $p = 1$ is of special interest, see [GMS99], [Koz98] and [Miy97]. In Section 5 we give an example of a function $f \in L^1(\mathbb{R}^{n+1}_+)^{n+1}$ satisfying $\operatorname{div} f = 0$ such that $u \notin L^1(\mathbb{R}^{n+1}_+)^{n+1}$, where $\hat{u}$ is the solution of the resolvent equation (2.1) defined below. This implies in particular that there is *no* estimate of the form

$$\|u(t,\cdot)\|_1 \leq C\|u_0\|_1 \tag{1.2}$$

for the solution of the Stokes Problem (1.1) with $f = 0$. This is remarkable because the solution of the Stokes equation on all of $\mathbb{R}^n$ satisfies an estimate of form (1.2). Taking into account however a recent result of Kozono [Koz98] on the equations of Navier-Stokes for exterior domains one does not expect an estimate of form (1.2) to hold true. Indeed, he showed that the existence of a local strong solution of the Navier-Stokes equations in L^1 implies that no force could act on the boundary of the domain which would mean that the Navier-Stokes equations are physically meaningless. Thus, from this point of view one does not expect that (1.2) holds true. However, the pointwise upper bound on the remainder term allows us to prove an estimate of the form

$$\|\nabla u(t,\cdot)\|_1 \leq C\frac{1}{t^{1/2}}\|u_0\|_1$$

for the solution of (1.1) with $f = 0$ which was proved first in [GMS99].

The pointwise upper bound on the remainder term allows us also to show that for $1 < p < \infty$ the Stokes operator admits a bounded H^∞-calculus on $L^p(H)^{n+1}$. In fact, even a stronger result is true: In Section 7 we prove that the Laplacian on $\mathbb{R}^n$ and the remainder term even admit an R-bounded H^∞-calculus on $L^p(H)^{n+1}$ for $1 < p < \infty$. Thus the Stokes operator admits an R-bounded H^∞-calculus on $L^p(H)^{n+1}$. As a consequence, we obtain maximal L^p–L^q-regularity for the solution of the Stokes equation (1.1).

For more information on the role of the Stokes semigroup in the theory of the Navier-Stokes equations, we refer to the recent article of Amann [Ama00] and the survey article by Wiegner [Wie99].

NOTATIONS AND CONVENTIONS. Throughout this paper, we define for $0 < \theta \leq \pi$ the sector Σ_θ in the complex plane by $\Sigma_\theta := \{z \in \mathbb{C}\backslash\{0\}; |\arg z| < \theta\}$. If X and Y are Banach spaces, $\mathcal{L}(X, Y)$ denotes the space of all bounded, linear operators from X to Y; moreover, $\mathcal{L}(X) := \mathcal{L}(X, X)$. The spectrum of a linear operator A in X is denoted by $\sigma(A)$. Given $p \in [1, \infty)$, we denote by

$$L^p_\sigma(H) := \{f \in C_c^\infty(H); \operatorname{div} f = 0\}^{-\|\cdot\|_{L^p}}$$

the Banach space of all solenoidal functions in $L^p(H)$. If $1 < p < \infty$, then

$$L^p(H) = L^p_\sigma(H) \oplus G^p(H),$$

where $G^p(H)$ consists of all functions $f \in L^p(H)$ for which there exists $g \in L^p_{loc}(H)$ such that $f = \nabla g$. The projection $P : L^p(H)$ onto $L^p_\sigma(H)$ is called the Helmholtz Projection.

By C, M and c we denote various constants which may differ from line to line, but which are always independent of the free variables.

2. The resolvent problem for the Stokes problem

In this section we consider the resolvent equation for the Stokes problem in the half space $H := \mathbb{R}^{n+1}_+ := \{(x, y) \in \mathbb{R}^{n+1};\ x \in \mathbb{R}^n,\ y > 0\}$. Given $p \in [1, \infty]$, $\lambda \in \sum_\pi$ and $f \in L^p(\mathbb{R}^{n+1}_+)^{n+1}$ with $\operatorname{div} f = 0$, find a velocity field $u = (u_1, \ldots, u_n, u_{n+1})^T$ and a pressure field p such that

$$\begin{aligned} \lambda u - \Delta u + \nabla p &= f \quad \text{in } H, \\ \operatorname{div} u &= 0 \quad \text{in } H, \\ u_{|\partial H} &= 0. \end{aligned} \tag{2.1}$$

For $x \in \mathbb{R}^n$ and $y > 0$ we write $u = (v, w)^T$ with $v = (v_1, \ldots, v_n)^T$ and $f = (f_v, f_w)^T$ with $f_v = ((f_v)_1, \ldots, (f_v)_n)^T$. Assume that $f_w(\xi, 0) = 0$ for all $\xi \in \mathbb{R}^n$. Applying the Fourier transform with respect to x we obtain

$$(\lambda + |\xi|^2)\hat{v}(\xi, y) - \partial_y^2 \hat{v}(\xi, y) = \hat{f}_v(\xi, y) - i\xi \cdot \hat{p}(\xi, y), \quad \xi \in \mathbb{R}^n, y > 0 \tag{2.2}$$

$$(\lambda + |\xi^2|)\hat{w}(\xi, y) - \partial_y^2 \hat{w}(\xi, y) = \hat{f}_w(\xi, y) - \partial_y \hat{p}(\xi, y), \quad \xi \in \mathbb{R}^n, y > 0 \tag{2.3}$$

$$i\xi \cdot \hat{v}(\xi, y) + \partial_y \hat{w}(\xi, y) = 0, \quad \xi \in \mathbb{R}^n, y > 0 \tag{2.4}$$

$$\hat{v}(\xi, 0) = 0, \quad \xi \in \mathbb{R}^n \tag{2.5}$$

$$\hat{w}(\xi, 0) = 0, \quad \xi \in \mathbb{R}^n. \tag{2.6}$$

Multiplying equation (2.2) by $i\xi$, applying ∂_y to (2.3) and adding them yields

$$\begin{aligned} &|\xi|^2 \hat{p}(\xi, y) - \partial_y^2 \hat{p}(\xi, y) = -\partial_y \hat{f}_w(\xi, y) - i\xi \hat{f}_v(\xi, y) \\ &\quad = -(\operatorname{div} f)\hat{\ }(\xi, y) = 0 \end{aligned}$$

for $\xi \in \mathbb{R}^n$ and $y > 0$, where we already took into account equation (2.4). Hence

$$\hat{p}(\xi, y) = e^{-|\xi| y} \widehat{p_0}(\xi), \quad \xi \in \mathbb{R}^n, \quad y > 0$$

for some function $\hat{p}_0$. We thus obtain for $\hat{v}$ and $\hat{w}$ the following representations

$$\begin{aligned} \hat{v}(\xi, y) &= \frac{1}{2\omega(\xi)} \int_0^\infty [e^{-\omega(\xi)|y-s|} - e^{-\omega(\xi)(y+s)}] \\ &\quad [\hat{f}_v(\xi, s) - i\xi e^{-|\xi| s} \hat{p}_0(\xi)]\, ds, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \hat{w}(\xi, y) &= \frac{1}{2\omega(\xi)} \int_0^\infty [e^{-\omega(\xi)|y-s|} - e^{-\omega(\xi)(y+s)}] \\ &\quad [\hat{f}_w(\xi, s) + |\xi| e^{-|\xi| s} \hat{p}_0(\xi)]\, ds, \end{aligned} \tag{2.8}$$

for $\xi \in \mathbb{R}^n$, $y > 0$ and where $\omega(\xi) := (|\lambda| + |\xi|^2)^{\frac{1}{2}}$ for $\xi \in \mathbb{R}^n$. In order to determine $\hat{p}_0$ consider $\partial_y \hat{w}(\xi, y)$ at $y = 0$, i.e.

$$\partial_y \hat{w}(\xi, 0) = \int_0^\infty e^{-\omega(\xi)s} [\hat{f}_w(\xi, s) + |\xi| e^{-|\xi| s} \hat{p}_0(\xi)]\, ds = 0, \quad \xi \in \mathbb{R}^n.$$

This implies that

$$\hat{p}_0(\xi) = -\frac{(\omega(\xi) + |\xi|)}{|\xi|} \int_0^\infty e^{-\omega(\xi)s} \hat{f}_w(\xi, s)\, ds, \quad \xi \neq 0. \tag{2.9}$$

By assumption, $i\xi \cdot \hat{f}_v(\xi, y) + \partial_y \hat{f}_w(\xi, y) = 0$ for $\xi \in \mathbb{R}^n$ and $y > 0$. Integrating by parts yields

$$\begin{aligned} -\omega(\xi) \int_0^\infty e^{-\omega(\xi)s} \hat{f}_w(\xi, s)\, ds &= \int_0^\infty \frac{d}{ds}(e^{-\omega(\xi)s}) \hat{f}_w(\xi, s)\, ds \\ &= \int_0^\infty e^{-\omega(\xi)s} (i\xi) \cdot \widehat{f_v}(\xi, s)\, ds. \end{aligned}$$

Thus

$$\hat{f}^L_w(\xi) := \int_0^\infty e^{-\omega(\xi)s} \hat{f}_w(\xi, s)\, ds = \\ -\frac{i\xi}{\omega(\xi)} \int_0^\infty e^{-\omega(\xi)s} \hat{f}_v(\xi, s)\, ds =: \frac{-i\xi}{\omega(\xi)} \hat{f}^L_v(\xi) \tag{2.10}$$

for all $\xi \in \mathbb{R}^n$. Inserting (2.9) and (2.10) into (2.8) and (2.7) we obtain

$$\begin{aligned} \hat{v} &= \hat{v}_1 + \hat{v}_2 \\ \hat{w} &= \hat{w}_1 + \hat{w}_2 \end{aligned} \tag{2.11}$$

with $\hat{v}_1, \hat{v}_2$ and $\hat{w}_1, \hat{w}_2$ given for $\xi \in \mathbb{R}^n$ and $y > 0$ by

$$\begin{aligned} \hat{v}_1(\xi, y) &= \frac{1}{2\omega(\xi)} \int_0^\infty [e^{-\omega(\xi)|y-s|} - e^{-\omega(\xi)(y+s)}] \hat{f}_v(\xi, s)\, ds \\ \hat{v}_2(\xi, y) &= \frac{1}{2\omega(\xi)} \int_0^\infty [e^{-\omega(\xi)|y-s|} - e^{-\omega(\xi)(y+s)}] \\ &\quad \frac{(i\xi)}{|\xi|} e^{-|\xi|s} (\omega(\xi) + |\xi|)\, ds\, \hat{f}^L_w(\xi) \\ &= \frac{i\xi}{|\xi|} \frac{1}{\omega(\xi) - |\xi|} [e^{-y|\xi|} - e^{-\omega(\xi)y}] \hat{f}^L_w(\xi) \\ &= \frac{1}{(w(\xi) - |\xi|)|\xi| w(\xi)} [e^{-y|\xi|} - e^{-w(\xi)y}](\xi \cdot \hat{f}^L_v(\xi))\xi \\ \hat{w}_1(\xi, y) &= \frac{1}{2\omega(\xi)} \int_0^\infty [e^{-\omega(\xi)|y-s|} - e^{-\omega(\xi)(y+s)}] \hat{f}_w(\xi, s)\, ds \\ \hat{w}_2(\xi, y) &= \frac{-1}{\omega(\xi) - |\xi|} [e^{-y|\xi|} - e^{-\omega(\xi)y}] \hat{f}^L_w(\xi) \\ &= \frac{i\xi}{\omega(\xi)} \cdot \frac{1}{\omega(\xi) - |\xi|} [e^{-y|\xi|} - e^{-\omega(\xi)y}] \hat{f}^L_v(\xi). \end{aligned}$$

Observe that

$$v_1 = (\lambda - \Delta_D)^{-1} f_v, \quad w_1 = (\lambda - \Delta_D)^{-1} f_w,$$

where Δ_D denotes the Laplacian in $\mathbb{R}_+^{n+1}$ subject to zero Dirichlet boundary conditions. It is well known that for $1 \le p \le \infty$ and $\lambda \in \Sigma_\theta$ with $\theta < \pi$ there exists a constant $M > 0$ such that

$$\begin{aligned} \|v_1\|_{L^p(\mathbb{R}_+^{n+1})^n} &\le \frac{M}{|\lambda|}\|f_v\|_{L^p(\mathbb{R}_+^{n+1})^n} \\ \|w_1\|_{L^p(\mathbb{R}_+^{n+1})} &\le \frac{M}{|\lambda|}\|f_w\|_{L^p(\mathbb{R}_+^{n+1})}. \end{aligned}$$

Hence, in order to obtain L^p-estimates for v and w we may restrict ourselves in the following to the cases v_2 and w_2. The L^p-estimates for v_2 and w_2 will be derived from pointwise upper bounds for the inverse Fourier transform of $\hat{v}_2$ and $\hat{w}_2$.

3. Pointwise upper bounds for the remainder term

We proved in the previous Section 2 that

$$\begin{aligned} v &= (\lambda - \Delta_D)^{-1} f_v + v_2 \\ w &= (\lambda - \Delta_D)^{-1} f_w + w_2 \end{aligned}$$

where v_2 and w_2 are defined as in (2.11). In this section we derive pointwise upper estimates for v_2 and w_2.

Let $\theta < \pi$ and define $\hat{r}_v : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+ \times \Sigma_\theta \to \mathbb{C}$ by

$$\hat{r}_v(\xi, y, y', \lambda) := \frac{e^{-|\xi|y} - e^{-\omega(\xi)y}}{\omega(\xi) - |\xi|} \frac{1}{|\xi|\omega(\xi)} \xi\xi^T e^{-\omega(\xi)y'}, \tag{3.1}$$

where $\omega(\xi) = \sqrt{|\lambda| + |\xi|^2}$. Define

$$r_v(x, y, y', \lambda) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{r}_v(\xi, y, y', \lambda) d\xi. \tag{3.2}$$

Note that r_v is well defined since the above integral is absolutely convergent for $(y, y') \ne (0, 0)$.

Observe first that by the following scaling argument it suffices to consider the case $|\lambda| = 1$ and $|\arg\lambda| \le \theta < \pi$. In fact, the change of coordinates

$$\xi \to |\lambda|^{\frac{1}{2}}\xi, \quad y \to \frac{y}{|\lambda|^{\frac{1}{2}}}, \quad y' \to \frac{y'}{|\lambda|^{\frac{1}{2}}}, \quad x \to \frac{x}{|\lambda|^{\frac{1}{2}}} \tag{3.3}$$

yield

$$\hat{r}_v(\xi, y, y', \lambda) = \frac{1}{|\lambda|^{\frac{1}{2}}} \hat{r}_v\left(\frac{\xi}{|\lambda|^{\frac{1}{2}}},\ |\lambda|^{\frac{1}{2}} y,\ |\lambda|^{\frac{1}{2}} y',\ \frac{\lambda}{|\lambda|}\right)$$

and hence

$$r_v(x, y, y', \lambda) = |\lambda|^{\frac{n-1}{2}} r_v\left(|\lambda|^{\frac{1}{2}}x,\ |\lambda|^{\frac{1}{2}}y,\ |\lambda|^{\frac{1}{2}}y',\ \frac{\lambda}{|\lambda|}\right).$$

For this reason, we may suppose now that $\lambda \in \Sigma_\theta$ with $|\lambda| = 1$.

For $z \in \mathbb{C}$ consider next the function ϕ given by

$$\phi(z) = \frac{1 - e^{-z}}{z}, \quad z \in \mathbb{C}\backslash\{0\}$$

and note that

$$|\phi(z)| \le \frac{C}{1 + |z|}, \quad \operatorname{Re} z \ge 0 \tag{3.4}$$

for a suitable constant $C > 0$. Thus

$$\hat{r}_v(\xi, y, y', \lambda) = ye^{-|\xi|y}e^{-\omega(\xi)y'}\frac{1}{|\xi|\omega(\xi)}\xi\xi^T\phi((\omega(\xi) - |\xi|)y).$$

Choose now a rotation Q in $\mathbb{R}^n$ such that $Qx = (|x|, 0, \dots, 0)$ and write

$$Q\xi = (a, rb), \quad a \in \mathbb{R}, \quad r > 0, \quad b \in \mathbb{R}^{n-1}, \quad |b| = 1.$$

By this coordinate transformation we obtain

$$\begin{aligned} r_v(x, y, y', \lambda) = c_n \int_0^\infty r^{n-2} \int_{S_{n-1}} \int_{-\infty}^\infty e^{i|x|a} \frac{y}{\sqrt{\lambda + r^2 + a^2}\sqrt{r^2 + a^2}} e^{-y\sqrt{r^2+a^2}} \\ \cdot e^{-y'\sqrt{\lambda+r^2+a^2}}\phi(y(\sqrt{\lambda + r^2 + a^2} - \sqrt{r^2 + a^2}))\begin{pmatrix} a \\ rb^T \end{pmatrix}(a\ rb)\, da\, db\, dr. \end{aligned} \tag{3.5}$$

where S_{n-1} is the unit sphere in $\mathbb{R}^{n-1}$.

Next, for $\varepsilon > 0$ small enough, we shift the path of integration for a to the contour $a \to s + i\varepsilon(r + |s|)$, $s \in \mathbb{R}$, without changing the value of the integral thanks to Cauchy's theorem. Then

$$r^2 + a^2 = r^2 + s^2 - \varepsilon^2(r + |s|)^2 + 2i\varepsilon(r + |s|)s.$$

Hence, given $\varepsilon_0 > 0$, there exists a constant $c > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ we have

$$c|r^2 + a^2| \le (r + |s|)^2 \le \frac{1}{c}|r^2 + a^2|, \quad r > 0, \quad s \in \mathbb{R}$$

and

$$|\arg(r^2 + a^2)| \le c\varepsilon, \quad r > 0, \quad s \in \mathbb{R}.$$

We thus obtain for $x \in \mathbb{R}^n$, $y > 0$, $y' > 0$ and $\lambda \in \Sigma_\theta$ with $|\lambda| = 1$ the bounds

$$\begin{aligned}
|e^{i|x|a}| &= e^{-\varepsilon|x|(r+|s|)},\\
|e^{-y\sqrt{r^2+a^2}}| &\leq e^{-cy(r+|s|)},\\
|e^{-y'\sqrt{|\lambda|+r^2+a^2}}| &\leq e^{-cy'(1+r+|s|)}, \quad |\lambda| = 1, \quad |\arg\lambda| \leq \theta < \pi,\\
\left|\frac{\sqrt{r^2+a^2}}{\sqrt{|\lambda|+r^2+a^2}}\right| &\leq c\,\frac{r+|s|}{1+r+|s|}\\
|\sqrt{|\lambda|+r^2+a^2} - \sqrt{r^2+a^2}| &= \frac{1}{\sqrt{|\lambda|+r^2+a^2}+\sqrt{r^2+a^2}} \geq c\,\frac{1}{1+r+|s|}.
\end{aligned}$$

Inserting these bounds into (3.5) yields for a multiindex α

$$\begin{aligned}
&|(\partial_x)^\alpha r_v(x, y, y', \lambda)|\\
&\quad \leq M \int_0^\infty r^{n-2} \int_0^\infty e^{-c(r+s)(|x|+y+y')} y e^{-cy'} \frac{(r+s)^{1+|\alpha|}}{1+r+s+y}\, dsdr\\
&\quad = Mye^{-cy'} \int_0^\infty r^{n-2} \int_r^\infty \frac{e^{-cs(|x|+y+y')}}{1+s+y} s^{1+|\alpha|}\, dsdr\\
&\quad = Mye^{-cy'} \int_0^\infty s^{n+|\alpha|} \frac{e^{-cs(|x|+y+y')}}{1+y+s}\, ds
\end{aligned}$$

for some constant $M > 0$ independent of $x \in \mathbb{R}^n$, $y, y' > 0$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $|\arg\lambda| \leq \theta < \pi$. We thus have proved the following result.

PROPOSITION 3.1. *Let $\theta \in (0, \pi)$ and α be a multiindex. Then there exist constants $M, c > 0$ such that*

$$|(\partial_x)^\alpha r_v(x, y, y', \lambda)| \leq Mye^{-cy'} \int_0^\infty \frac{s^{n+|\alpha|}}{1+y+s} e^{-cs(|x|+y+y')}\, ds,$$

where $x \in \mathbb{R}^n$, $y, y' > 0$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $|\arg\lambda| \leq \theta < \pi$.

REMARK 3.2. For $\theta < \pi$ we define $\hat{r}_w : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+ \times \Sigma_\theta \to \mathbb{C}^n$ by

$$\hat{r}_w(\xi, y, y', \lambda) := \frac{e^{-|\xi|y} - e^{-\omega(\xi)y}}{\omega(\xi) - |\xi|} \frac{i\xi}{\omega(\xi)} e^{-\omega(\xi)y'} \tag{3.6}$$

Copying the above proof we see that the assertion of Proposition 3.1 remains true if r_v is replaced by r_w, where

$$r_w(x, y, y', \lambda) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{r}_w(\xi, y, y', \lambda) d\xi, \quad x \in \mathbb{R}^n, y, y' > 0. \tag{3.7}$$

The kernel estimates given in Proposition 3.1 and Remark 3.2 allow us to derive L^p-estimates for v_2 and w_2 via the following lemma on L^p-continuity of integral operators acting in half spaces.

LEMMA 3.3. *Suppose that* $1 \le p \le \infty$ *and let* $\frac{1}{p} + \frac{1}{p'} = 1$. *Let* T *be an integral operator in* $L^p(\mathbb{R}^{n+1}_+)$ *of the form*

$$(Tf)(x, y) = \int_0^\infty \int_{\mathbb{R}^n} k(x - x', y, y') f(x', y') dx' dy', \quad x \in \mathbb{R}^n, y > 0,$$

where $k : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C}$ *is a measurable function.*

a) *Let* $1 < p < \infty$. *If*

$$\left(\int_0^\infty \left(\int_0^\infty \|k(\cdot, y, y')\|_1^{p'} dy' \right)^{\frac{p}{p'}} dy \right)^{\frac{1}{p}} =: M_1 < \infty,$$

then $T \in \mathcal{L}(L^p(\mathbb{R}^{n+1}_+))$ *and* $\|T\|_{\mathcal{L}(L^p(\mathbb{R}^{n+1}_+))} \le M_1$.

b) *Let* $p = 1$. *If*

$$\sup_{y'>0} \int_0^\infty \|k(\cdot, y, y')\|_1 \, dy =: M_2 < \infty,$$

then $T \in \mathcal{L}(L^1(\mathbb{R}^{n+1}_+))$ *and* $\|T\|_{\mathcal{L}(L^1(\mathbb{R}^{n+1}_+))} \le M_2$.

c) *Let* $p = \infty$. *If*

$$\sup_{y>0} \int_0^\infty \|k(\cdot, y, y')\|_1 \, dy' =: M_3 < \infty,$$

then $T \in \mathcal{L}(L^\infty(\mathbb{R}^{n+1}_+))$ *and* $\|T\|_{\mathcal{L}(L^\infty(\mathbb{R}^{n+1}_+))} \le M_3$.

Proof. a) By Young's inequality and Hölder's inequality we have

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} |Tf(x,y)|^p dx dy \\
&= \int_0^\infty \left\| \int_0^\infty (k(\cdot, y, y') * f(\cdot, y'))(\cdot) dy' \right\|_p^p dy \\
&\le \int_0^\infty \left(\int_0^\infty \|k(\cdot, y, y')\|_1 \|f(\cdot, y')\|_p \, dy' \right)^p dy \\
&\le \|f\|^p_{L^p(\mathbb{R}^{n+1}_+)} \int_0^\infty \left(\int_0^\infty \|k(\cdot, y, y')\|_1^{p'} \right)^{\frac{p}{p'}} dy.
\end{aligned}$$

The assertions b), c) are proved in a simliar way. □

Combining the estimates obtained in Proposition 3.1 and Remark 3.2 with Lemma 3.3 we obtain the following estimates for v_2 and w_2.

PROPOSITION 3.4. *Let* $1 < p \le \infty$ *and* $\theta \in (0, \pi)$. *Let* v_2 *and* w_2 *be defined as in* (2.11). *Then there exists a constant* $M > 0$ *such that*

$$\begin{aligned}
\|v_2\|_{L^p(\mathbb{R}^{n+1}_+)^n} &\le \frac{M}{|\lambda|} \|f_v\|_{L^p(\mathbb{R}^{n+1}_+)^n} \\
\|w_2\|_{L^p(\mathbb{R}^{n+1}_+)} &\le \frac{M}{|\lambda|} \|f_v\|_{L^p(\mathbb{R}^{n+1}_+)^n}
\end{aligned}$$

for all $\lambda \in \Sigma_\theta$.

Proof. Observe that for $\lambda \in \Sigma_\theta$

$$v_2(x,y) = \int_0^\infty \int_{\mathbb{R}^n} r_v(x - x', y, y', \lambda) f_v(x', y') dx' dy', \quad x \in \mathbb{R}^n, y > 0,$$

where

$$r_v(x, y, y', \lambda) = |\lambda|^{\frac{n-1}{2}} r_v \left(|\lambda|^{\frac{1}{2}} x, |\lambda|^{\frac{1}{2}} y, |\lambda|^{\frac{1}{2}} y', \frac{\lambda}{|\lambda|} \right), \quad x \in \mathbb{R}^n, y, y' > 0$$

and r_v is satisfying the estimate given in Proposition 3.1. In order to prove the assertion, we verify the conditions of Lemma 3.3 a) and c).

To this end, note that

$$\begin{aligned}
\int_{\mathbb{R}^n} |(\partial_x)^\alpha r_v(x,y,y',\lambda)|dx &= |\lambda|^{\frac{n-1}{2}} \int_{\mathbb{R}^n} \left|(\partial_x)^\alpha r_v\left(|\lambda|^{\frac12}x, |\lambda|^{\frac12}y, |\lambda|^{\frac12}y', \frac{\lambda}{|\lambda|}\right)\right| dx \\
&\le C_n |\lambda|^{\frac{n-1}{2}} |\lambda|^{\frac12} y e^{-c|\lambda|^{\frac12}y'} \int_0^\infty\int_0^\infty \frac{s^{n+|\alpha|}\rho^{n-1}}{1+|\lambda|^{\frac12}y+s} e^{-cs|\lambda|^{\frac12}(\rho+y+y')} d\rho ds \\
&\le C_n |\lambda|^{\frac{n}{2}} y e^{-c|\lambda|^{\frac12}y'} \int_0^\infty \frac{s^{|\alpha|}}{1+|\lambda|^{\frac12}y+s}\frac{1}{|\lambda|^{\frac{n}{2}}} e^{-cs|\lambda|^{\frac12}(y+y')} ds \qquad (3.8)\\
&\le C_n y e^{-c|\lambda|^{\frac12}y'} \int_0^\infty \frac{e^{-cs|\lambda|^{\frac12}(y+y')}}{1+|\lambda|^{\frac12}y} s^{|\alpha|} ds \\
&\le C_n e^{-c|\lambda|^{\frac12}y'} \frac{y}{1+|\lambda|^{\frac12}y} \cdot \frac{1}{(|\lambda|^{\frac12}(y+y'))^{1+|\alpha|}}.
\end{aligned}$$

Hence, if $1 < p < \infty$ and $|\alpha| = 0$, then

$$\int_0^\infty \|r_v(\cdot,y,y',\lambda)\|_1^{p'} dy' \le C_{n,p} \frac{1}{|\lambda|^{\frac12}} \cdot \frac{1}{|\lambda|^{\frac{p'}{2}}} \frac{1}{(1+|\lambda|^{\frac12}y)^{p'}}$$

and

$$\begin{aligned}
&\int_0^\infty \left(\int_0^\infty \|r_v(\cdot,y,y',\lambda)\|_1^{p'} dy'\right)^{\frac{p}{p'}} dy \\
&\quad \le C_{np} \frac{1}{|\lambda|^{\frac{p}{2p'}}} \cdot \frac{1}{|\lambda|^{\frac{p}{2}}} \frac{1}{|\lambda|^{\frac12}} \int_0^\infty \frac{1}{(1+\sigma)^p} d\sigma = C_{np}\left(\frac{1}{|\lambda|}\right)^p.
\end{aligned}$$

If $p = \infty$, then

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} |r_v(\cdot,y,y',\lambda)| dx dy' \\
&\quad \le C_n \int_0^\infty e^{-c|\lambda|^{\frac12}y'} dy' \cdot \frac{y}{1+|\lambda|^{\frac12}y} \cdot \frac{1}{|\lambda|^{\frac12}y} \le C_n \frac{1}{|\lambda|} \frac{1}{1+|\lambda|^{\frac12}y}
\end{aligned} \qquad (3.9)$$

and hence

$$\sup_{y>0} \int_0^\infty \|r_v(\cdot,y,y',\lambda)\|_1 dy' \le \frac{C_n}{|\lambda|}$$

The estimate for r_w follows in exactly the same way. □

Note that the kernel estimate given in Proposition 3.1 does not allow to verify the assumptions of Lemma 3.3 b) for the case $p = 1$. We investigate this point in detail in Section 5. Summing up we proved the following result:

THEOREM 3.5. *Let* $1 < p \leq \infty$, $0 < \theta < \pi$ *and* $\lambda \in \Sigma_\theta$. *Let* $f \in L^p(\mathbb{R}^{n+1}_+)^{n+1}$ *such that div* $f = 0$ *and* $f_{n+1|_{\partial H}} = 0$. *Let* $u = (v, w)^T$ *be defined as in* (2.11). *Then there exists a constant* $M > 0$ *such that*

$$\|u\|_{L^p(\mathbb{R}^{n+1}_+)^{n+1}} \leq \frac{M}{|\lambda|}\|f\|_{L^p(\mathbb{R}^{n+1}_+)^{n+1}}.$$

For $1 < p < \infty$ and $\lambda > 0$ consider the mapping

$$R(\lambda) : L^p_\sigma(H) \to L^p_\sigma(H), \quad f \mapsto u_\lambda,$$

where u_λ is defined as in (2.11). Let A be the Stokes operator in $L^p_\sigma(H)$ defined as in Section 1. Then $R(\lambda)(\lambda - A)f = f$ for all $f \in D(A)$ and $(\lambda - A)R(\lambda)f = f$ for all $f \in L^p_\sigma(H)$. Thus

$$R(\lambda) = (\lambda - A)^{-1}, \quad \lambda > 0.$$

Theorem 3.5 implies now the following well known result (see e.g. [Sol77], [McC81], [Gig81], [FS96], [BS87], [Uka87], [GS89]).

COROLLARY 3.6. *Let* $1 < p < \infty$. *Then the Stokes operator* A *defined as in Section* 1 *generates an analytic strongly continuous semigroup on* $L^p_\sigma(H)^{n+1}$.

4. The Stokes operator in $BUC_\sigma(H)$, $C_{0,\sigma}(H)$, $L^\infty_\sigma(H)$

In this section we define the Stokes operator in $BUC_\sigma(H)$, $C_{0,\sigma}(H)$, $L^\infty_\sigma(H)$ and show that it is the generator of an analytic semigroup on these spaces (which is not strongly continuous in the case of $L^\infty_\sigma(H)$). To this end, define

$$BUC_\sigma(H) := \{f \in BUC(H); \operatorname{div} f = 0,\ f(x_1, \dots, x_n, 0) = 0$$
$$\text{for all} \quad x_1, \dots, x_n \in \mathbb{R}\}$$

and

$$C_{0,\sigma}(H) := \overline{\{f \in C_c^\infty(H); \operatorname{div} f = 0\}}^{\|\cdot\|_{L^\infty}}.$$

Let $X_\sigma(H)$ be one of the spaces $BUC_\sigma(H)$ or $C_{0,\sigma}(H)$. For $\theta \in (0, \pi)$ and $\lambda \in \Sigma_\theta$ consider the mapping

$$R(\lambda) : X_\sigma(H)^{n+1} \to L^\infty(H)^{n+1}, \quad f \mapsto u_\lambda,$$

where u_λ is the solution of the Stokes equation given in (2.11). Theorem 3.5 and a direct calculation show that $\{R(\lambda); \lambda > 0\}$ is a pseudo-resolvent in $X_\sigma(H)^{n+1}$.

LEMMA 4.1. *Let* $f \in X_\sigma(H)^{n+1}$. *Then*

$$\lim_{\lambda\to\infty} \lambda R(\lambda) f = f.$$

Proof. Notice first that

$$\begin{aligned} R(\lambda) f_v &= (\lambda - \Delta_D)^{-1} f_v + v_2(\lambda) \\ R(\lambda) f_w &= (\lambda - \Delta_D)^{-1} f_w + w_2(\lambda), \end{aligned}$$

where Δ_D denotes the Dirichlet Laplacian and v_2, w_2 are defined as in (2.11). Since Δ_D generates a C_0-semigroup on $BUC(H)$ or $C_0(H)$, respectively, it follows that

$$\lim_{\lambda\to\infty} \lambda(\lambda - \Delta_D)^{-1} f = f$$

for all $f \in X_\sigma(H)$. It thus remains to prove that $\lim_{\lambda\to\infty} \lambda v_2(\lambda) = 0$ in $X_\sigma(H)^n$ and $\lim_{\lambda\to\infty} \lambda w_2(\lambda) = 0$ in $X_\sigma(H)$.

In order to do so, note first that $\lim_{\lambda\to\infty} \lambda v_2(\lambda) = 0$ in $BUC_\sigma(H)^n$ if and only if

$$\begin{aligned} \lim_{\lambda\to 0} \sup_{y>0} \Big| \int_0^\infty \int_{\mathbb{R}^n} & \lambda^{(n+1)/2} r_v(\lambda^{1/2}x', \lambda^{1/2}y, \lambda^{1/2}y', 1) \\ & f_v(x - x', y') dx' dy' \Big| = 0. \end{aligned} \tag{4.1}$$

Since

$$\int_{\mathbb{R}^n} \lambda^{(n+1)/2} r_v(\lambda^{1/2}x', \lambda^{1/2}y, \lambda^{1/2}y', 1)\, dx' = \lambda \hat{r_v}(0, y, y', \lambda) = 0$$

it follows that (4.1) is satisfied provided

$$\begin{aligned} \lim_{\lambda\to 0} \sup_{y>0} \int_0^\infty \int_{\mathbb{R}^n} & |\lambda^{(n+1)/2} r_v(\lambda^{1/2}x', \lambda^{1/2}y, \lambda^{1/2}y', 1)| \\ & |f_v(x - x', y') - f_v(x, 0)| dx' dy' = 0. \end{aligned}$$

But

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^n} |\lambda^{(n+1)/2} r_v(\lambda^{1/2}x', y, \lambda^{1/2}y', 1)| |f_v(x - x', y') - f_v(x, 0)| dx' dy' \\ & = \int_0^\infty \int_{\mathbb{R}^n} |r_v(x', y, y', 1)| \left| f_v\left(x - \frac{x'}{\lambda^{1/2}}, \frac{y'}{\lambda^{1/2}}\right) - f_v(x, 0) \right| dx' dy' \\ & \le \int_0^\infty \int_{\mathbb{R}^n} |r_v(x', y, y', 1)| dx' dy' \cdot \sup_{|x'|\le R, |y'|\le S} \\ & \quad \left| f_v\left(x - \frac{x'}{\lambda^{1/2}}, \frac{y'}{\lambda^{1/2}}\right) - f_v(x, 0) \right| \end{aligned}$$

$$+\left[\int_S^\infty \int_{\mathbb{R}^n} |r_v(x', y, y', 1)| dx' dy' \right.$$
$$\left. + \int_0^\infty \int_{|x'|\geq R} |r_v(x', y, y', 1)| dx' dy' \right] \cdot 2|f_v|_\infty$$
$$\leq M \sup_{|x'|\leq \frac{R}{\lambda^{1/2}}, |y'|\leq \frac{S}{\lambda^{1/2}}} |f_v(x-x', y') - f_v(x,0)| + 2|f_v|_\infty M \left(\frac{1}{S} + \frac{1}{R}\right)$$

for all $S, R > 0$ by (3.9) and since

$$\int_S^\infty \int_{\mathbb{R}^n} |r_v(x', y, y', 1)| dx' dy' \leq M \int_S^\infty \frac{e^{-cy'}}{y'} dy' \leq \frac{M}{S}$$

by (3.8) and

$$\int_0^\infty |r_v(x', y, y', 1)| dy' \leq M \int_0^\infty e^{-cs(|x'|+y)} \frac{s^n}{1+y+s} \frac{y}{c+s} ds$$
$$\leq M \int_0^\infty e^{-cs|x'|} s^n ds$$

by the estimate given in Proposition 3.1 which implies that

$$\int_0^\infty \int_{|x'|\geq R} |r_v(x', y, y', 1)| dx' dy' \leq \frac{M}{R}.$$

This implies the assertion if $X_\sigma(H) = BUC_\sigma(H)$. The case where of $X_\sigma(H) = C_{0,\sigma}(H)$ is proved in a similar way. □

The above lemma shows that $\ker R(\lambda) = 0$ for all $\lambda > 0$. Hence, there exists a closed, densely defined operator A_{X_σ} in $X_\sigma(H)^{n+1}$ such that

$$R(\lambda) = (\lambda - A_{X_\sigma})^{-1}, \lambda > 0.$$

DEFINITION 4.2. The operator A_{X_σ} is called the *Stokes operator in* $X_\sigma(H)^{n+1}$.

Theorem 3.5 implies now the following result.

THEOREM 4.3. *The Stokes operator A_{X_σ} generates a strongly continuous analytic semigroup on $X_\sigma(H)^{n+1}$.*

Finally, we consider the solution of the Stokes equation in $L^\infty_\sigma(H)$. This space is defined as follows: Observe that ∇ acts as a bounded operator from $\hat{W}^{1,1}(H)$ into $L^1(H)^{n+1}$, where $\hat{W}^{1,1}(H) = \{f \in L^1_{loc}(H); \nabla f \in L^1(H)\}$. Hence $Div := -\nabla^*$ is a bounded operator from $L^\infty(H)^{n+1}$ into $\hat{W}^{1,1}(H)^*$. We define

$$L^\infty_\sigma(H)^{n+1} := \ker Div.$$

Thus $f \in L^\infty_\sigma(H)$ if and only if $\int_H \nabla\varphi f = 0$ for all $\varphi \in W^{1,1}(H)$. Consider the mapping $R(\lambda) : L^\infty_\sigma(H)^{n+1} \to L^\infty(H)^{n+1}$ defined as before. Theorem 3.5 and a direct calculation implies that $\{R(\lambda); \lambda > 0\}$ is a pseudo-resolvent. In contrast to the situation of $X_\sigma(H)$ we do *not* have that $\lim_{\lambda\to\infty} \lambda v_2(\lambda) = 0$ in $L^\infty_\sigma(H)$. However, the representation of the remainder term given in Section 3 allows us to show that $\ker R(\lambda) = 0$ in $L^\infty_\sigma(H)$ for all $\lambda > 0$. Thus there exists a closed operator $A_{L^\infty_\sigma}$ in $L^\infty_\sigma(H)^{n+1}$ such that

$$R(\lambda) = (\lambda - A_{L^\infty_\sigma})^{-1}, \lambda > 0.$$

We call the operator $A_{L^\infty_\sigma}$ the *Stokes operator in* $L^\infty_\sigma(H)^{n+1}$. Note that $A_{L^\infty_\sigma}$ is not densely defined; however Theorem 3.5 implies the following result.

THEOREM 4.4. *The Stokes operator* $A_{L^\infty_\sigma}$ *generates an analytic semigroup on* $L^\infty_\sigma(H)^{n+1}$ *(which is not strongly continuous in* 0*).*

5. The case $p = 1$

In this section we give an example of a function $f \in L^1(\mathbb{R}^{n+1}_+)^{n+1}$ satisfying $\operatorname{div} f = 0$ such that $w_2 \notin L^1(\mathbb{R}^{n+1}_+)$ where w_2 is defined as in Section 2. This is rather surprising since $u \in L^p(\mathbb{R}^{n+1}_+)^{n+1}$ whenever $f \in L^p(\mathbb{R}^{n+1}_+)^{n+1}$ with $\operatorname{div} f = 0$ for all $p \in (1,\infty]$ as we have seen in Section 2. More precisely, we have the following result.

THEOREM 5.1. *Let* $0 < \theta < \pi$ *and* $\lambda \in \Sigma_\theta$. *Then there exists* $f = (f_1, \dots, f_{n+1}) \in L^1(\mathbb{R}^{n+1}_+)^{n+1}$ *satisfying* $\operatorname{div} f = 0$ *and* $f_{n+1}(x, 0) = 0$ *for all* $x \in \mathbb{R}^n$ *such that* $u \notin L^1(\mathbb{R}^{n+1}_+)^{n+1}$, *where* $\hat{u} = (\hat{v}, \hat{w})$ *defined as in* (2.11) *is the solution of the resolvent equation* (2.2)–(2.6).

We base the construction of our counterexample on well known properties of the Hardy space $H^1(\mathbb{R}^n)$. We remind the reader that

$$H^1(\mathbb{R}^n) := \{f \in L^1(\mathbb{R}^n) : f^* \in L^1(\mathbb{R}^n)\}$$

where f^* is given by

$$f^*(x) := \sup_{t>0} |(k_t * f)(x)|, \quad x \in \mathbb{R}^n,$$

and k_t denotes the Gaussian kernel given by $k_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$ $(x \in \mathbb{R}^n,\ t > 0)$. Equipped with the norm $\|f\|_{H^1(\mathbb{R}^n)} := \|f^*\|_{L^1(\mathbb{R}^n)}$, the space $H^1(\mathbb{R}^n)$ becomes a Banach space. It is well known that an L^1-function f belongs to $H^1(\mathbb{R}^n)$ if and only if its Riesz transforms $R_j f$ belong to $L^1(\mathbb{R}^n)$ for all $j \in \{1, \dots, n\}$. This property of $H^1(\mathbb{R}^n)$ will be of crucial importance for the following.

Proof of Theorem 5.1. Consider the Gaussian kernel k_r for some $r > 0$. Then $k_r \notin H^1(\mathbb{R}^n)$ because $k_r^* \notin L^1(\mathbb{R}^n)$. Thus, since $k_r \in L^1(\mathbb{R}^n)$, there exists $j \in \{1, \ldots, n\}$ such that $R_j k_r \notin L^1(\mathbb{R}^n)$. Fix $j \in \{1, \ldots, n\}$ with this property and define for $x \in \mathbb{R}^n$, $y > 0$ and $r > 0$ the function $f_w : \mathbb{R}_+^{n+1} \to \mathbb{R}$ by

$$f_w(x, y) := 4yT_{\omega_1}(y)\frac{\partial}{\partial x_j}(1 - \Delta)^{3/2}k_r(x), \quad x \in \mathbb{R}^n, \quad y > 0.$$

Here T_{w_1} denotes the C_0-semigroup on $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$ generated by the pseudo-differential operator A given by $Af = -\mathcal{F}^{-1}(\omega_1 \hat{f})$, where $\omega_1(\xi) = \sqrt{1 + |\xi|^2}$ for $\xi \in \mathbb{R}^n$. Note that $\|T_{\omega_1}(y)\| \leq Me^{-y}$ for suitable $M > 0$ and all $y \geq 0$. Hence

$$\int_0^\infty \|f_w(\cdot, y)\|^p_{L^p(\mathbb{R}^n)}\, dy \leq C \int_0^\infty y^p e^{-yp}\, dy < \infty,$$

for suitable $C > 0$, which means that $f_w \in L^p(\mathbb{R}_+^{n+1})$ for all $p \in [1, \infty)$. Observe moreover that

$$\hat{f}_w(\xi, y) = 4ye^{-\omega_1(\xi)y}(i\xi_j)\omega_1^3(\xi)e^{-r|\xi|^2}, \quad \xi \in \mathbb{R}^n, y > 0$$

and

$$\partial_y \hat{f}_w(\xi, y) = (i\xi_j)4\omega_1^3(\xi)e^{-r|\xi|^2}(e^{-\omega_1(\xi)y} - \omega_1(\xi)ye^{-\omega_1(\xi)y}) =: i\xi_j \hat{g}_r(\xi, y)$$

with $g_r \in L^p(\mathbb{R}_+^{n+1})$ for $1 \leq p < \infty$. Set now $f := (f_v, f_w)^T$ with $f_v := (0, \ldots, -g_r, 0, \ldots, 0)$ so that the j-th component of f_v is $-g_r$. Thus $\operatorname{div} f = 0$, $f \in L^p(\mathbb{R}_+^{n+1})^{n+1}$ and $(f \cdot \nu)_{|\partial H} = 0$, where ν denotes the outer unit normal. Observe next that the functions

$$\begin{aligned}
\xi \mapsto \hat{f}_w^L(\xi) &= 4\int_0^\infty se^{-2\omega_1(\xi)s}(i\xi_j)\omega_1^3(\xi)e^{-r|\xi|^2}ds = (i\xi_j)e^{-r|\xi|^2}\omega_1(\xi)\\
\xi \mapsto \hat{w}_2(\xi, s) &= -(i\xi_j)\omega_1(\xi)e^{-r|\xi|^2}(\omega_1(\xi) + |\xi|)[e^{-s|\xi|} - e^{-\omega_1(\xi)s}], \quad s > 0\\
\xi \mapsto \int_0^\infty \hat{w}_2(\xi, s)ds &= -\frac{1}{|\xi|\omega_1(\xi)}(i\xi_j)\omega_1(\xi)e^{-r|\xi|^2} = -\frac{1}{2}\frac{i\xi_j}{|\xi|}e^{-r|\xi|^2}
\end{aligned}$$

all belong to $L^2(\mathbb{R}^n)$. It thus follows from Plancherel's theorem that

$$\int_0^\infty w_2(x, y)dy = -\mathcal{F}^{-1}(r_j(\cdot)e^{-r|\cdot|^2})(x),$$

where $r_j(\xi) = i\frac{\xi_j}{|\xi|}$ for $\xi \neq 0$. Hence

$$\begin{aligned}
\|w_2\|_{L^1(\mathbb{R}^{n+1}_+)} &= \int\limits_{\mathbb{R}^n} \int\limits_0^\infty |w_2(x,y)|\, dydx \\
&\geq \int\limits_{\mathbb{R}^n} \left| \int\limits_0^\infty w_2(x,y)dy \right| dx \\
&= \int\limits_{\mathbb{R}^n} |\mathcal{F}^{-1}(r_j(\cdot)e^{-r|\cdot|^2})(x)|\, dx \\
&= \|R_j k_r\|_{L^1(\mathbb{R}^n)} = \infty,
\end{aligned}$$

by the choice of j at the beginning of this section. This implies the assertion.

REMARK 5.2. It is an open problem to decide whether the assertion of Theorem 5.1 remains true if the half space H is replaced by a *bounded* domain with smooth boundary.

We now turn our attention to gradient estimates in the L^1-norm of the solution of the Stokes equation as they were proved in [GMS99] in 1999. To this end let $|\lambda| = 1$ and note that by (3.8) we have for $|\alpha| = 1$

$$\begin{aligned}
&\sup_{y'>0} \int_0^\infty \int_{\mathbb{R}^n} |(\partial_x)^\alpha r_v(x,y,y',\lambda)| dxdy \\
&\quad \leq C \int_0^\infty \int_0^\infty \frac{sy}{1+y+s} e^{-csy} dsdy \\
&\quad = 2C \int_0^\infty \int_0^y \frac{sy}{1+y+s} e^{-csy} dsdy < \infty.
\end{aligned}$$

Moreover, note that

$$\begin{aligned}
&\partial_y \hat{r_v}(\xi,y,y',\lambda) = -e^{-\omega(\xi)y'} e^{-|\xi|y} \frac{\xi\xi^T}{\omega(\xi)} \\
&\quad \frac{1-e^{-(\omega(\xi)-|\xi|)y}}{\omega(\xi)-|\xi|} + e^{-\omega(\xi)(y+y')} \frac{\xi\xi^T}{\omega(\xi)|\xi|}.
\end{aligned}$$

The first term on the right hand side above is estimated exactly in the same way as $\partial_x r_v$. In order to treat the second term denote its inverse Fourier transform by r_{v1}. For $|\lambda| = 1$, we obtain

$$|r_{v1}(x,y,y',\lambda)| \leq M \int_0^\infty r^{n-2} \int_0^\infty e^{-c|x|(r+s)} e^{-c(y+y')(1+r+s)} \frac{r+s}{1+r+s} dsdr$$

Thus

$$\sup_{y'>0}\int_0^\infty\int_{\mathbb{R}^n}|r_{v1}(x,y,y',\lambda)|dxdy\leq M\int_0^\infty\int_0^\infty\frac{1}{1+s}e^{-cy(1+s)}dsdy<\infty.$$

Lemma 3.3 b), a scaling argument and a simliar argument for r_w implies now the following result which was proved first in [GMS99].

PROPOSITION 5.3. *There exists a constant $M>0$ such that the solution of the Stokes equation* (1.1) *with right hand side $f=0$ satisfies*

$$\|\nabla u(t,\cdot)\|_1\leq\frac{M}{t^{1/2}}\|u_0\|_1,\qquad t>0.$$

6. Bounded H^∞-calculus

Given $\theta\in(0,\pi)$, we denote by $H^\infty(\Sigma_\theta)$ the Banach algebra of all bounded holomorphic functions $f:\Sigma_\theta\to\mathbb{C}$ equipped with the supremum norm. We also denote by $H_0^\infty(\Sigma_\theta)$ the set of all $g\in H^\infty(\Sigma_\theta)$ such that there exist constants $C\geq 0$, $s>0$ with

$$|g(z)|\leq C\frac{|z|^s}{1+|z|^{2s}},\quad z\in\Sigma_\theta.$$

Let now $\omega\in(0,\pi)$ and let A be a closed, densely defined operator in a Banach space X which is one to one and has dense range. Assume that $\sigma(A)\subset\overline{\Sigma_\omega}$ and that for every $\omega'\in(\omega,\pi)$ there exists $M>0$ such that

$$\|(\lambda-A)^{-1}\|\leq\frac{M}{|\lambda|},\quad \lambda\in\mathbb{C}\backslash\overline{\Sigma_{\omega'}}.$$

Let $\omega<\theta<\pi$. Then, given $g\in H_0^\infty(\Sigma_\theta)$, the operator

$$g(A):=\frac{1}{2\pi i}\int_\Gamma g(\lambda)(\lambda-A)^{-1}d\lambda$$

is a well defined element of $\mathcal{L}(X)$, where Γ denotes the positively oriented contour $\{\lambda=te^{\pm i\omega'};\,t\geq 0\}$ for some $\omega'\in(\omega,\theta)$. Moreover, for $z\in\Sigma_\vartheta$ set $h(z):=z(1+z)^{-2}$. Then

$$f(A):=[h(A)]^{-1}(fh)(A),\quad f\in H^\infty(\Sigma_\vartheta)$$

is a well defined operator in X. Let $0<\omega<\theta<\pi$. We say that A *admits a bounded H^∞-calculus on the sector* Σ_θ if $f(A)\in\mathcal{L}(X)$ and there exists a constant $M>0$ such that

$$\|f(A)\|_{\mathcal{L}(X)}\leq M\|f\|_\infty,\quad f\in H^\infty(\Sigma_\vartheta).\tag{6.1}$$

It is well known, that A satisfies (6.1) if there exists $M > 0$ such that

$$\|g(A)\|_{\mathcal{L}(X)} \leq M\|g\|_\infty, \quad g \in H_0^\infty(\Sigma_\vartheta).$$

For details see [CDMY96]. Thus in deriving estimates for $f(A)$, it suffices to establish estimates for $g(A)$, where $g \in H_0^\infty(\Sigma_\vartheta)$.

For $1 < p < \infty$ let A be the Stokes operator in $L^p(H)^{n+1}$ defined as in Section 1. We show in the following that $-A$ admits a bounded H^∞-calculus on $L^p(H)^{n+1}$ for every sector Σ_θ with $0 < \theta < \pi$. We start with the following lemma.

LEMMA 6.1. *Let T be an integral operator of the form*

$$(Tf)(y) = \int_0^\infty k(y, y')f(y')dy', \quad y > 0, \tag{6.2}$$

where $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C}$ is a measurable function such that the above integral is well defined. Suppose that for some $p \in (1, \infty)$ there exists a constant $M > 0$ such that

$$|(Tf)(y)| \leq \frac{M}{y^{\frac{1}{p}}}\|f\|_{L^p(\mathbb{R}_+)}, \quad y > 0.$$

If $T \in \mathcal{L}(L^{q_0}(\mathbb{R}_+))$ for some $q_0 \in (p, \infty]$, then $T \in \mathcal{L}(L^q(\mathbb{R}_+))$ for all $q \in (p, q_0]$.

Proof. By assumption, Tf is dominated pointwise by a function belonging to the weak L^p-space $L^p_w(\mathbb{R}_+)$. Thus $T : L^p(\mathbb{R}_+) \to L^p_w(\mathbb{R}_+)$ is a bounded operator. The assumption and the Marcinkiewicz interpolation theorem imply that $T \in \mathcal{L}(L^q(\mathbb{R}_+))$ for all $q \in (p, q_0]$. □

COROLLARY 6.2. *Let $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C}$ be a measurable function. Suppose that there exists $M > 0$ such that*

$$|k(y, y')| \leq \frac{M}{y + y'} \log\left(1 + \frac{y}{y'}\right), \quad y, y' > 0.$$

Let T be defined as in (6.2) and let $1 < p \leq \infty$. Then $T \in \mathcal{L}(L^p(\mathbb{R}_+))$.

Proof. Note first that

$$\int_0^\infty |k(y, y')|dy' \leq M \int_0^\infty \frac{\log(1 + \frac{y}{y'})}{1 + \frac{y}{y'}}\frac{dy'}{y'} = M\int_0^\infty \frac{\log(1+s)}{(1+s)s}ds < \infty$$

which implies that $T \in \mathcal{L}(L^\infty(\mathbb{R}_+))$.

If $p > 1$ let $\frac{1}{p} + \frac{1}{p'} = 1$. For $f \in L^p(\mathbb{R}_+)$ we obtain by Hölder's inequality

$$\begin{aligned} |Tf(y)| &\leq M \left(\int_0^\infty \frac{\log^{p'}(1 + \frac{y}{y'})dy'}{(y+y')^{p'}} \right)^{\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}_+)} \\ &= \frac{M}{y^{\frac{1}{p}}} \int_0^\infty \frac{\log^{p'}(1+s)}{(1+s)^{p'}} \frac{1}{s^{2-p'}} ds \|f\|_{L^p(\mathbb{R}_+)} \\ &\leq \frac{M}{y^{\frac{1}{p}}} \|f\|_{L^p(\mathbb{R}_+)}, \quad y > 0. \end{aligned}$$

Thus the assertion follows from Lemma 6.1. ☐

Let now $h \in H_0^\infty(\Sigma_\theta)$ where $0 < \theta < \pi$ is fixed. Consider the function

$$k_{h,v}(x, y, y') = \frac{1}{2\pi i} \int_\Gamma h(\lambda) r_v(x, y, y', -\lambda) d\lambda, \quad x \in \mathbb{R}^n, y, y' > 0,$$

where r_v is defined as in (3.2) and $\Gamma := \{\rho e^{\pm i\varphi}, \rho \geq 0\}$ with $0 < \varphi < \theta$. The estimate for r_v given in Proposition 3.1 yields

$$\begin{aligned} |k_{h,v}(x, y, y')| &\leq C\|h\|_{H^\infty} \int_0^\infty |r_v(x, y, y', \rho e^{\pm i(\pi - \varphi)})| d\rho \\ &\leq C\|h\|_{H^\infty} \int_0^\infty \rho^{\frac{n}{2}} y e^{-c\rho^{\frac{1}{2}} y'} \int_0^\infty \frac{s^n e^{-c\rho^{\frac{1}{2}} s(|x|+y+y')}}{1 + \rho^{\frac{1}{2}} y + s} ds d\rho \\ &\leq C\|h\|_{H^\infty} \int_0^\infty y e^{-c\rho^{\frac{1}{2}} y'} \int_0^\infty \sigma^n \frac{e^{-c\sigma(|x|+y+y')}}{\rho^{\frac{1}{2}} + \rho y + \sigma} d\sigma d\rho \\ &=: C\|h\|_{H^\infty} k_1(x, y, y'). \end{aligned} \tag{6.3}$$

Now

$$\begin{aligned} \int_{\mathbb{R}^n} |k_1(x, y, y')| dx &\leq C_n \int_0^\infty y e^{-c\rho^{\frac{1}{2}} y'} \int_0^\infty \frac{e^{-c\sigma(y+y')}}{\rho^{\frac{1}{2}} + \rho y + \sigma} d\sigma d\rho \\ &\leq C \frac{y}{y+y'} \int_0^\infty \frac{e^{-c\rho^{\frac{1}{2}} y'}}{\rho^{\frac{1}{2}} + \rho y} d\rho \\ &= C \frac{y}{y+y'} \int_0^\infty \frac{e^{-csy'}}{1+sy} ds, \quad y, y' > 0. \end{aligned}$$

Splitting the latter integral at $s = \frac{1}{y'}$, we obtain

$$\begin{aligned}\int_0^\infty \frac{ye^{-csy'}}{1+sy}ds &\le \int_0^{\frac{1}{y'}} \frac{y}{1+sy}ds + \frac{y}{1+\frac{y}{y'}}\int_{\frac{1}{y'}}^\infty e^{-csy'}ds \\ &\le \log\left(1+\frac{y}{y'}\right) + C\frac{\frac{y}{y'}}{1+\frac{y}{y'}} \le C\log\left(1+\frac{y}{y'}\right).\end{aligned}$$

This implies that

$$\int_{\mathbb{R}^n} |k_1(x,y,y')|dx \le \frac{C}{y+y'}\log\left(1+\frac{y}{y'}\right), \quad y, y' > 0. \tag{6.4}$$

Define now the operator $T_{h,v}$ in $L^p(\mathbb{R}^{n+1}_+)$ by

$$(T_{h,v}f)(x,y) := \int_0^\infty \int_{\mathbb{R}^n} k_{h,v}(x-x',y,y')f(x',y')dx'dy', \quad x \in \mathbb{R}^n, y > 0. \tag{6.5}$$

Then, by Young's inequality, (6.3), (6.4) and Corollary 6.2 we have

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n} |(T_{h,v}f)(x,y)|^p dxdy &\le \int_0^\infty \left(\int_0^\infty \|k_{h,v}(\cdot,y,y')\|_1 \|f(\cdot,y')\|_p \, dy'\right)^p dy \\ &\le C\|h\|^p_{H^\infty} \int_0^\infty \left(\int_0^\infty \frac{1}{y+y'}\log\left(1+\frac{y}{y'}\right)\|f(\cdot,y')\|_p \, dy'\right)^p dy \\ &\le C\|h\|^p_{H^\infty}\|f\|^p_{L^p(\mathbb{R}^{n+1}_+)}.\end{aligned} \tag{6.6}$$

Moreover, by Remark 3.2 the function r_w defined as in (3.7) satisfies also an estimate of the form given in Proposition 3.1. Thus, the function $k_{h,w}$ defined by

$$k_{h,w}(x,y,y') := \frac{1}{2\pi i}\int_\Gamma h(\lambda) r_w(x,y,y',-\lambda)d\lambda, \quad x \in \mathbb{R}^n, y, y' > 0$$

also satisfies

$$|k_{h,w}(x,y,y')| \le C\|h\|_{H^\infty} k_1(x,y,y'), \quad x \in \mathbb{R}^n, y, y' > 0.$$

Define the operator $T_{h,w}$ as in (6.5) with $k_{h,v}$ replaced by $k_{h,w}$. We conclude that $T_{h,w}$ satisfies estimate (6.6).

Finally note that the operator $-\Delta_D$ admits a bounded $H^\infty(\Sigma_\theta)$-calculus on $L^p(\mathbb{R}^{n+1}_+)$ for every $\theta \in (0,\pi)$. This follows from Corollary 7.3 below or by the results proved in [PS93]. This fact and (6.6) imply the following result.

THEOREM 6.3. *Let* $1 < p < \infty$ *and let* A *be the Stokes operator in defined as in* $L^p(\mathbb{R}_+^{n+1})^{n+1}$. *Then* $-A$ *admits a bounded* $H^\infty(\Sigma_\theta)$*-calculus on* $L^p(\mathbb{R}_+^{n+1})^{n+1}$ *for each* $\theta \in (0, \pi)$.

As a consequence we obtain by the Dore-Venni theorem maximal L^p–L^q-estimates for the solution of the Stokes equation (see also [GS91], [Gig85]).

7. R-bounded H^∞-calculus for $-\Delta$ on $\mathbb{R}^{n+1}$ and $\mathbb{R}_+^{n+1}$

R-bounded families of operators play an important role in the variant of the Dore-Venni theorem which has been recently proved by Kalton and Weis [KW00]. They are defined as follows: let X and Y be Banach spaces. We call a family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ *R-bounded*, if there is a constant $C > 0$ and $p \in [1, \infty)$ such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and all independent, symmetric $\{-1, 1\}$-valued random variables ε_j on a probability space $(\Omega, \mathcal{M}, \mu)$ we have

$$\left\| \sum_{j=1}^N \varepsilon_j T_j x_j \right\|_{L^p(\Omega;Y)} \le C \cdot \left\| \sum_{j=1}^N \varepsilon_j x_j \right\|_{L^p(\Omega;X)} .$$

The smallest such C is called the *R-bound* of $\mathcal{T}$ and is denoted by $R(\mathcal{T})$. For detailed information on this subject and its relation to maximal L^p-regularity and to L^p-Fourier multipliers, we refer to [CdPSW00], [Wei99] and [CP00].

REMARK 7.1. a) Let X and Y be Hilbert spaces. Then $\mathcal{T} \subset \mathcal{L}(X; Y)$ is R-bounded if and only if $\mathcal{T}$ is uniformly bounded.

b) Let $1 \le p < \infty$ and let $X = Y = L^p(G)$ for some $G \subset \mathbb{R}^n$ open. Then $\mathcal{T} \subset \mathcal{L}(X, Y)$ is R-bounded if and only if there is a constant $M > 0$ such that the following square function estimate

$$\left\| \left(\sum_{j=1}^N |T_j f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(G)} \le M \left\| \left(\sum_{j=1}^N |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(G)} \tag{7.1}$$

holds for each $N \in \mathbb{N}$, $f \in L^p(G)$ and $T_j \in \mathcal{T}$.

This assertion is a coinsequence of *Khintchine's inequality*. For $p \in [1, \infty)$ there exists a constant $K_p > 0$ such that

$$K_p^{-1} \left\| \sum_{j=1}^N \varepsilon_j a_j \right\|_{L^p(\Omega)} \le \left(\sum_{j=1}^N |a_j|^2 \right)^{\frac{1}{2}} \le K_p \left\| \sum_{j=1}^N \varepsilon_j a_j \right\|_{L^p(\Omega)}$$

for all $N \in \mathbb{N}, a_j \in \mathbb{C}$ and all independent, symmetric $\{-1,1\}$-valued random variables ε_j on $(\Omega, \mathcal{M}, \mu)$.

Let $\Omega \subset \mathbb{R}^n$ open and let $1 < p < \infty$. Assume that a given operator A in $L^p(\Omega)$ admits a bounded H^∞-calculus on $L^p(\Omega)$ for some sector Σ_θ with $0 < \theta < \pi$. By the recent results given in [KW00] it is an interesting question to ask whether or not the set

$$\{h(A) : h \in H_0^\infty(\Sigma_\theta), \|h\|_{H^\infty(\Sigma_\theta)} \le R\} \subset \mathcal{L}(L^p(\Omega)) \tag{7.2}$$

is R-bounded. If this holds true, we say that A admits an *R-bounded H^∞-calculus on $L^p(\Omega)$ for the sector Σ_θ.*

Our first result in this section asserts that $-\Delta$ on $L^p(\mathbb{R}^n)$, where Δ denotes the Laplacian, admits an R-bounded H^∞-calculus on $L^p(\mathbb{R}^n)$ for each sector Σ_θ where $0 < \theta < \pi$. More precisely, the following holds true.

THEOREM 7.2. *Let $1 < p < \infty$ and denote by Δ the Laplacian in $\mathbb{R}^n$. Then $-\Delta$ admits an R-bounded H^∞-calculus on $L^p(\mathbb{R}^n)$ on the sector Σ_θ for $0 < \theta < \pi$.*

Proof. Let $h \in H_0^\infty(\Sigma_\theta)$, where $0 < \theta < \pi$. Then the Fourier transform of $h(-\Delta)$ is given by $h(|\xi|^2)$ for $\xi \in \mathbb{R}^n$. The kernel $k_h(\cdot)$ corresponding to $h(|\cdot|^2)$ is given by

$$k_h(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\xi} h(|\xi|^2) d\xi, \quad x \in \mathbb{R}^n.$$

Choosing a rotation Q such that $Qx = (|x|, 0, \ldots, 0)$ and writing $Q\xi = (a, rb)$, with $a \in \mathbb{R},\ r > 0,\ b \in \mathbb{R}^{n-1},\ |b| = 1$ we obtain

$$k_h(x) = c_n \int_0^\infty r^{n-2} \int_{-\infty}^\infty h(r^2 + a^2) e^{i|x|a} da dr, \quad x \in \mathbb{R}^n.$$

Next we deform the contour of integration via Cauchy's theorem to

$$a = s + i\epsilon(r + |s|), \quad r > 0, s \in \mathbb{R}.$$

We then obtain

$$\left| \frac{\operatorname{Im}(r^2 + a^2)}{\operatorname{Re}(r^2 + a^2)} \right| = \frac{2\varepsilon|s|(r + |s|)}{r^2 + s^2 - \varepsilon^2(r + |s|)^2} \le \frac{4\varepsilon(r^2 + s^2)}{(1 - 2\varepsilon^2)(r^2 + s^2)} \le C\varepsilon,$$

which implies that for ε small enough our new contour stays inside Σ_θ. Thus

$$\begin{aligned}|k_h(x)| &\leq C\int_0^\infty r^{n-2}\int_0^\infty \|h\|_{H^\infty} e^{-c|x|(r+s)}dsdr\\ &\leq C\|h\|_{H^\infty}\int_0^\infty |x|^{1-n}e^{-c|x|s}ds = C\frac{\|h\|_{H^\infty}}{|x|^n}, \quad x\in\mathbb{R}^n\backslash\{0\}.\end{aligned}$$

Similarly,

$$|D^\alpha k_h(x)| \leq C_\alpha\|h\|_{H^\infty}\frac{1}{|x|^{n+|\alpha|}}, \quad x\in\mathbb{R}^n\backslash\{0\}$$

for each multiindex α. If $|x|\geq 2|y|$ we obtain

$$\begin{aligned}|k_h(x-y)-k_h(x)| &= \left|\int_0^1 \frac{d}{dt}k_h(x-ty)dt\right|\\ &\leq |y|\int_0^1 \frac{dt}{|x-ty|^{n+1}}\|h\|_{H^\infty} \leq C\frac{|y|}{|x|^{n+1}}\|h\|_{H^\infty}.\end{aligned}$$

This implies that for $R>0$ the uniform Hörmander condition is satisfied, i.e.:

$$\begin{aligned}&\int_{|x|>2|y|}\sup_{\|h\|_{H^\infty}\leq R}|k_h(x-y)-k_h(x)|dx\\ &\leq C|y|R\int_{|x|>2|y|}\frac{dx}{|x|^{n+1}} = C|y|R\int_{2|y|}^\infty\frac{dr}{r^2} = CR.\end{aligned} \tag{7.3}$$

Since $A=-\Delta$ is selfadjoint in $L^2(\mathbb{R}^n)$ we see that the set

$$\{h(A) : h\in H_0^\infty(\Sigma_\theta),\ \|h\|_{H^\infty(\Sigma_\theta)}\leq R\}\subset\mathcal{L}(L^2(\mathbb{R}^n))$$

is uniformly bounded. By Remark 7.1 a) and 7.1 b) it follows that there is a constant $C>0$ such that

$$\left\|\left(\sum_{j=1}^N |H_j f_j|^2\right)^{\frac{1}{2}}\right\|_{L^2(\mathbb{R}^n)} \leq CR\left\|\left(\sum_{j=1}^N |f_j|^2\right)^{\frac{1}{2}}\right\|_{L^2(\mathbb{R}^n)}$$

for $N \in \mathbb{N}$, $H_j := h_j(A)$, $h_j \in H_0^\infty(\Sigma_\theta)$, $\|h_j\|_{H^\infty(\Sigma_\theta)} \leq R$. Set now $X := l_2^N(\mathbb{N})$ and define $K : L^p(\mathbb{R}^n, X) \to L^p(\mathbb{R}^n, X)$ by

$$(Kf)_i := H_i f_i, \quad i = 1, \ldots, N.$$

The uniform Hörmander condition (7.3) implies that

$$\int\limits_{|x|>2|y|} \|K(x-y) - K(x)\|\, dx \leq CR.$$

Since K acts as a bounded operator on $L^2(\mathbb{R}^n; X)$, the Benedek-Calderon-Panzone theorem (see e.g. [Hie99]) implies that K is L^p-bounded for $1 < p < \infty$. This means that there is a constant $C > 0$ such that

$$\left\| \left(\sum_{j=1}^N |H_j f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \left(\sum_{j=1}^N |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}.$$

Remark 7.1 b) implies that the set

$$\{h(A) : h \in H_0^\infty(\Sigma_\theta),\ \|h\|_{H^\infty(\Sigma_\theta)} \leq R\} \subset \mathcal{L}(L^p(\mathbb{R}^n))$$

is R-bounded for all $R > 0$ and all $0 < \theta < \pi$. □

COROLLARY 7.3. *Let $1 < p < \infty$. Denote by Δ_D the Dirichlet Laplacian in $\mathbb{R}_+^{n+1}$. Then $-\Delta_D$ admits an R-bounded $H^\infty(\Sigma_\theta)$-calculus on $L^p(\mathbb{R}_+^{n+1})$ for each $\theta \in (0, \pi)$.*

Proof. The resolvent of the Dirichlet Laplacian Δ_D in $\mathbb{R}_+^{n+1}$ may be written as

$$(\lambda + \Delta_D)^{-1} = P_0(\lambda + \Delta)^{-1} E_0 - P_0 R(\lambda + \Delta)^{-1} E_0, \quad \lambda \in \mathbb{C}\backslash\Sigma_\theta,$$

where Δ denotes the Laplacian in $\mathbb{R}^{n+1}$, E_0 denotes extension by 0 to $\mathbb{R}^{n+1}$, P_0 the projection from $\mathbb{R}^{n+1}$ to $\mathbb{R}_+^{n+1}$ and R the reflection in the normal coordinate y. Therefore we have for $h \in H^\infty(\Sigma_\theta)$

$$h(-\Delta_D) = P_0 h(-\Delta) E_0 - P_0 R h(-\Delta) E_0.$$

The assertion thus follows from Theorem 7.2. □

We finally turn our attention to the remainder term of the Stokes operator as defined in Section 5. We start with the following proposition.

PROPOSITION 7.4. *Let* $1 \leq p < \infty$, $G \subset \mathbb{R}^n$ *be open and let* $\mathcal{T} = \{T_\mu : \mu \in M\} \subset \mathcal{L}(L^p(\Omega))$ *be a family of integral operators of the form*

$$(T_\mu f)(x) = \int_G k_\mu(x, y) f(y) dy, \quad x \in G, f \in L^p(G),$$

such that there exists a function k_0 *with*

$$|k_\mu(x, y)| \leq k_0(x, y), \quad f.a.a.\, x, y \in G \in M.$$

For $x \in G$ *set* $(T_0 f)(x) = \int_G k_0(x, y) f(y) dy$. *If* $T_0 \in \mathcal{L}(L^p(G))$, *then* $\mathcal{T} \subset \mathcal{L}(L^p(G))$ *is* R*-bounded.*

Proof. Due to Remark 7.1 b) it suffices to verify the square function estimate (7.1). But this follows easily from the L^p-boundedness of the dominating operator T_0. □

Combining (6.5), (6.6) and Corollary 6.2 with Proposition 7.4 we obtain the following result.

COROLLARY 7.5. *Let* $1 < p < \infty, 0 < \theta < \pi$. *Let* $k_{h,v}$ *and* $k_{h,w}$ *be defined as in Section* 5. *For* $h \in H_0^\infty(\Sigma_\theta)$ *define operators* $T_{h,v}$ *and* $T_{h,w}$ *as in* (6.5). *Let* $R > 0$. *Then the sets*

$$\{T_{h,v} : h \in H_0^\infty(\Sigma_\theta), \|h\|_{H^\infty(\Sigma_\theta)} \leq R\} \subset \mathcal{L}(L^p(\mathbb{R}_+^{n+1}))$$
$$\{T_{h,w} : h \in H_0^\infty(\Sigma_\theta), \|h\|_{H^\infty(\Sigma_\theta)} \leq R\} \subset \mathcal{L}(L^p(\mathbb{R}_+^{n+1}))$$

are R*-bounded.*

Summarizing, we proved the following result.

THEOREM 7.6. *Let* $1 < p < \infty$. *Denote by* A *the Stokes operator in* $L^p_\sigma(\mathbb{R}_+^{n+1})$. *Then* $-A$ *admits an* R*-bounded* $H^\infty(\Sigma_\theta)$*-calculus on* $L^p_\sigma(\mathbb{R}_+^{n+1})$ *for each* $\theta \in (0, \pi)$.

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